Computing Stability Criteria of Liquid Layers Spread over a Segment of a Sphere

Choong Yun Cho*
Forest Service, U. S. Department of Agriculture, Madison, Wis.

The recent paper by Anliker and Beam[1] develops the stability of liquid layers spread over simple curved bodies and leads to an infinite set of linear algebraic equations as the stability criteria for the case of a sphere. The coefficients of the linear equations consist of integrals of the product of two associated Legendre functions of degrees which have to be determined from the boundary condition. It is the task of this paper to show recurrence relations among the coefficients to make it feasible to use a high-speed computer for a further study of the equation.

According to Ref. 1, the following equation should be set up, dropping the index m except for L and M:

$$\sigma \frac{F_j(a)}{\dot{F}_j(a)} L_{jj}{}^m h_j + \frac{T}{\rho a^2}[2 - p_j(p_j+1)]L_{jj}{}^m h_j - g\sum_{n=1}^{\infty} M_{jn}{}^m h_n = 0$$

and determine if $\Delta(\sigma) = 0$, where $\Delta(\sigma)$ is the determinant of the coefficients, and h_n are the unknowns ($j = 1, 2, \ldots$; $m = 0, 1, 2, \ldots$).

To unify formulas, adopt the following definitions:

$$P_n(x) = (1/2^n n!)(d^n/dx^n)(x^2-1)^n$$
$$P_n{}^m(x) = (-1)^m (1-x^2)^{m/2}(d^m/dx^m)P_n(x) \qquad m \geq 0$$

for the Legendre polynomials and associated Legendre functions, respectively. First find k such that, for a given m,

$$(d/d\theta)P_k{}^m(\cos\theta)|_{\theta=\alpha} = 0 \qquad (1)$$

where α is one half of the angle of a segment of a sphere (see Ref. 1). From the formulas[2-4]

$$(d/d\theta)P_k{}^m(x) = (m+k)(m-k-1)P_k{}^{m-1}(x) - m\cot\theta\, P_k{}^m(x) \qquad (2)$$

$$(2k+1)\sin\theta\, P_k{}^m(x) = P_{k-1}{}^{m+1}(x) - P_{k+1}{}^{m+1}(x) \qquad (3)$$

$$(2k+1)\,x\,P_k{}^m(x) = (k-m+1)P_{k+1}{}^m(x) + (k+m)P_{k-1}{}^m(x) \qquad (4)$$

where $x = \cos\theta$, Eq. (1) becomes

$$(k+2)(m+k+1)P_k{}^m(\cos\alpha) + (k+1)(m-k-2)P_{k+2}{}^m(\cos\alpha) = 0 \qquad (5)$$

Now Eq. (5) can be solved for k by constructing $P_k{}^m(\cos\alpha)$ by a recurrence formula [e.g., Eq. (4)] for a given m. Denote such k's by $k_n (n = 1, 2, 3, \ldots)$.

To obtain $M_{jn}{}^m$ it can be seen that

$$(2k_j+1)M_{jn}{}^m = (k_j - m + 1)G^m_{k_j+1,k_n} + (k_j+m)G^m_{k_j-1,k_n} \qquad (6)$$

where

$$M_{jn}{}^m = \int_{\cos\alpha}^1 x\, P_{k_j}{}^m(x)P_{k_n}{}^m(x)\, dx \qquad (7)$$

$$G_{s,t}{}^m = \int_{\cos\alpha}^1 P_s{}^m(x)\, P_t{}^m(x)\, dx \qquad (8)$$

and $G^m_{k_j,k_n} = 0$ if $j \neq n$ (see Ref. 2). It can be shown[5] that for any integers, $n \geq m \geq 0$, $i \geq m \geq 0$, one has

$$(2i+1)E_{i,m}{}^m(x) = (2m-1)(i+m-1)(i+m)E^{m-1}_{i-1,m-1}(x) - (2m-1)(i-m+2)(i-m+1)E^{m-1}_{i+1,m-1}(x) \qquad (9)$$

$$(n-m)(2i+1)E_{i,n}{}^m(x) = (2n-1)(i-m+1)E^m_{i+1,n-1}(x) + (2n-1)(i+m)E^m_{i-1,n-1}(x) - (2i+1)(n+m-1)E^m_{i,n-2}(x) \qquad (10)$$

where

$$E_{i,n}{}^m(x) = \int P_i{}^m(x)\, P_n{}^m(x)\, dx \qquad (11)$$

and $E_{i,n}{}^m = 0$ if $n < m$ or $i < m$. Therefore,

$$G_{s,t}{}^m = E_{s,t}{}^m(1) - E_{s,t}{}^m(\cos\alpha)$$

For a given m, $E_{i,m}{}^m(x)$ can be constructed readily by Eq. (9), using the starting polynomials $E_{i,0}{}^0(x) = \int P_i(x)\, dx$, and then one can proceed to build $E_{i,n}{}^m(x)$ by Eq. (10).

For $L_{jj}{}^m$, it is seen at once that $L_{jj}{}^m = G^m_{k_j,k_j}$. The cases of hemispheres and spheres correspond to $\alpha = \pi/2$ and $\alpha = \pi$, respectively.

Received January 2, 1963.
* Consulting Mathematician, Forest Products Laboratory.

References

[1] Anliker, M. and Beam, R. M., "On the stability of liquid layers spread over simple curved bodies," J. Aerospace Sci. 29, 1196–1209 (1962).
[2] Robin, L., *Fonctions Sphériques de Legendre et Fonctions Sphéroïdales* (Gauthier-Villars, Paris, 1958), Vol. II, p. 195.
[3] Barnett, M. P., "The expansion and integration of products of surface harmonics and of certain related quantities," Theoretical Chemistry Inst. Rept. WIS-ONR-30, Univ. Wis. (March 1958).
[4] Hobson, E. W., *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge University Press, London, 1931), p. 108.
[5] Cho, C. Y., "Integrals of the product of two associated Legendre functions," (to be published).

Water Impact of the Mercury Capsule: Correlation of Analysis with NASA Tests

J. D. Rosenbaum* and W. R. Jensen*
Grumman Aircraft Engineering Corporation, Bethpage, N. Y.

Reference 1 is essentially an extension of the von Kármán approach for rotationally constrained prismatic bodies impacting smooth water. The method presented accounts for the behavior of a pitching nonprismatic vehicle penetrating rough water. The mathematical model is roughly similar to that for seaplane hulls.

In the absence of a more suitable method, the procedure proposed in Ref. 1 has been applied to the water landings of the Apollo command module. To establish the applicability of the technique to such a configuration, a comparison of predicted results for the Mercury capsule with experimental values from NASA[2] has been made. The results of this correlation are presented herein.

The mathematical model employed in the procedure is that shown in Fig. 1. The spherical bottom of the Mercury capsule is represented by a series of wedges, each with a 10° deadrise angle. Because of the constant deadrise, the chine heights

Received January 7, 1963.
* Structural Methods Engineer.

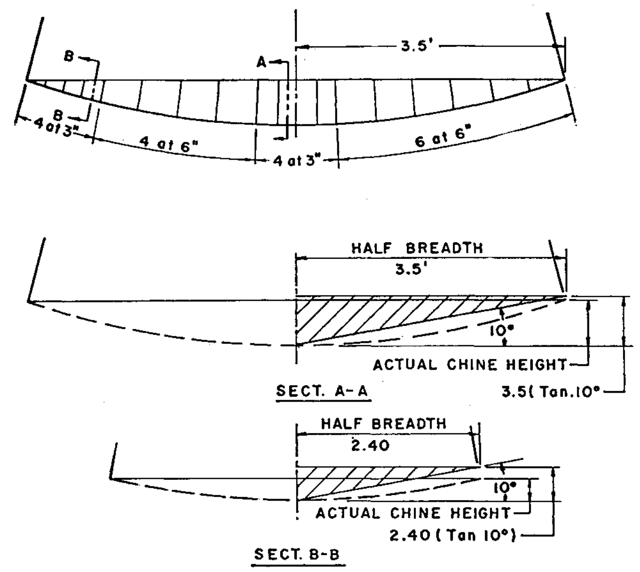

Fig. 1 Wedge configuration of the hull bottom. A 10° deadrise angle is assumed for each wedge, and the corresponding chine height is adjusted to comply with the half-breadth

are adjusted using the actual beam of the capsule. This retains the circular section in planform. The thickness of the wedges is reduced in the areas near the points of initial impact to decrease the effect of immersion of an individual wedge.

It will be noted, when considering Figs. 2 and 3 that, although the peak values have been attained satisfactorily, the acceleration onset rate lags the experimental results. This characteristic is inherent in the prediction, since a fixed deadrise angle, independent of immersion depth, is used. The fixed deadrise angle (10°) should be employed as a lower limit, since the theoretical results, using the Wagner coefficient for wedge impacts, are uncertain for smaller angles.

The steps in the curves are caused by the abrupt immersion of a finite wedge element (rise) and the abrupt immersion of a

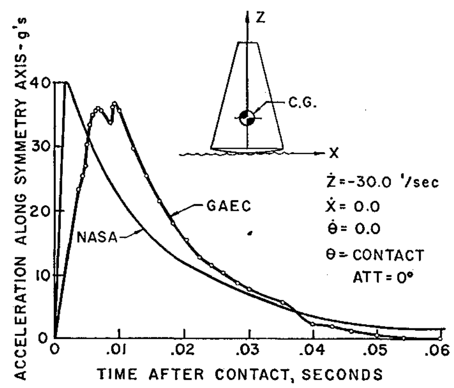

Fig. 2 Correlation for contact attitude of 0°

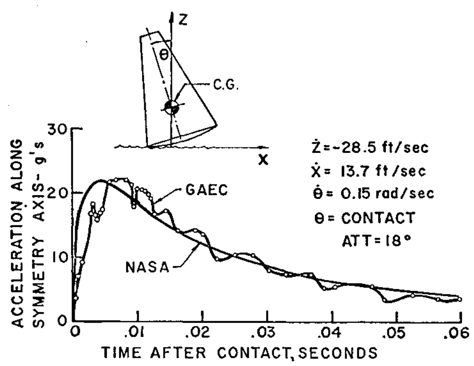

Fig. 3 Correlation for contact attitude of 18°

chine (drop). This effect can be reduced by a finer subdivision of wedges.

It must be noted that the method should be used primarily as a first approximation, basically for design purposes. When used in this way, the value of the procedure is evident. Any advances in the technique to increase accuracy must be accompanied by considerable theoretical advances in the field of either wedge immersion or immersion of other shapes.

References

[1] Mueller, W. H. and Malakoff, J. L., "Water impact of manned spacecraft," Grumman Aircraft Engineering Corp., Rept. ADR 04-03a-61.1 (April 1961); also ARS J. **31,** 1751–1760 (1961).

[2] McGhee, J. R., Hathaway, M. E., and Vaughan, V. L., Jr., "Water landing characteristics of a re-entry capsule," Langley Research Center, Langley Field, Va., NASA Memo. 5-23-59L (May 23, 1959).

Class of Exact Solutions of Nonisentropic, One-Dimensional Magnetohydrodynamic Flow

Roy M. Gundersen*

Illinois Institute of Technology, Chicago, Ill.

With a suitable choice of dependent variables, problems in one-dimensional unsteady gasdynamics may be reduced to finding solutions of a particular Monge-Ampère partial differential equation.[1, 2] If the particle paths and isobars coincide, this technique fails, so that an alternative procedure must be employed. This particular class of flows was the subject of a recent paper by Weir,[3] and it is the purpose of the present paper to show that Weir's discussion may be extended to magnetohydrodynamic flows subjected to a transverse magnetic field and that the resultant solutions reduce, in the limit of vanishing magnetic field, to those obtained by Weir.

The one-dimensional unsteady motion of an ideal, inviscid, perfectly conducting, compressible fluid subjected to a transverse magnetic field, i.e., the induction $\mathbf{B} = (0,0,B)$, is governed by the system of equations[4]

$$\rho_t + \rho u_x + \rho_x u = 0 \tag{1}$$

$$\rho(u_t + uu_x) + P_x + BB_x/\mu = 0 \tag{2}$$

$$B_t + uB_x + Bu_x = 0 \tag{3}$$

$$s_t + us_x = 0 \tag{4}$$

$$P = \exp[(s - s^*)/c_v]\rho^\gamma \tag{5}$$

where $\rho,u,P,s,s^*,\mu,b^2 = B^2/\mu\rho$, and γ are, respectively, the density, particle velocity, pressure, specific entropy, specific entropy at some reference state, permeability, square of the Alfvén speed, and ratio of specific heat at constant pressure c_p and at constant volume c_v. Partial derivatives are denoted by subscripts, and all dependent variables are functions of x and t alone.

The characteristics of this system are given by

$$dx/dt = u,\ u + \omega,\ u - \omega \tag{6}$$

where $\omega = [b^2 + c^2]^{1/2}$, the true speed of sound, is the limiting case of a fast wave, and c is the local speed of sound. Further, $\rho[dx - u\,dt]$ is the exact differential of a function ψ,

Received December 17, 1962.

* Associate Professor of Mathematics.